How Low
Can the Energy Density Go?

Aron Wall

Stanford Institute for Theoretical Physics

It from Qubit
2019 → Cambridge

Einstein Field Equation:

$$R_{\mu\nu} - (1/2)g_{\mu\nu}R \; = \; 8\pi G\,T_{\mu\nu}$$

curvature of spacetime

stress-energy tensor
of matter fields

Stress-Energy Tensor

the stress-energy tensor is a 4x4 symmetric matrix;

can be interpreted in a “local inertial coordinate system” (t, x, y, z) as:

$$T_{\mu\nu} = \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \left(\begin{array}{c|ccc} T_{tt} & T_{tx} & T_{ty} & T_{tz} \\ \hline \text{“} & T_{xx} & T_{xy} & T_{xz} \\ \text{“} & \text{“} & T_{yy} & T_{yz} \\ \text{“} & \text{“} & \text{“} & T_{zz} \end{array}\right) \end{array}$$

Stress-Energy Tensor

the stress-energy tensor is a 4x4 symmetric matrix;

can be interpreted in a “local inertial coordinate system” (t, x, y, z) as:

$$T_{\mu\nu} = \begin{array}{c|ccc} t & x & y & z \\ \text{energy density} & \multicolumn{3}{c}{\text{momentum-density (= energy flux)}} \\ \hline \text{“} & \text{x-pressure} & & \text{stress} \\ \text{“} & \text{“} & \text{y-pressure} & \\ \text{“} & \text{“} & \text{“} & \text{z-pressure} \end{array}$$

Spacetime geometry is not fixed *a priori*
—what spacetimes are allowed?

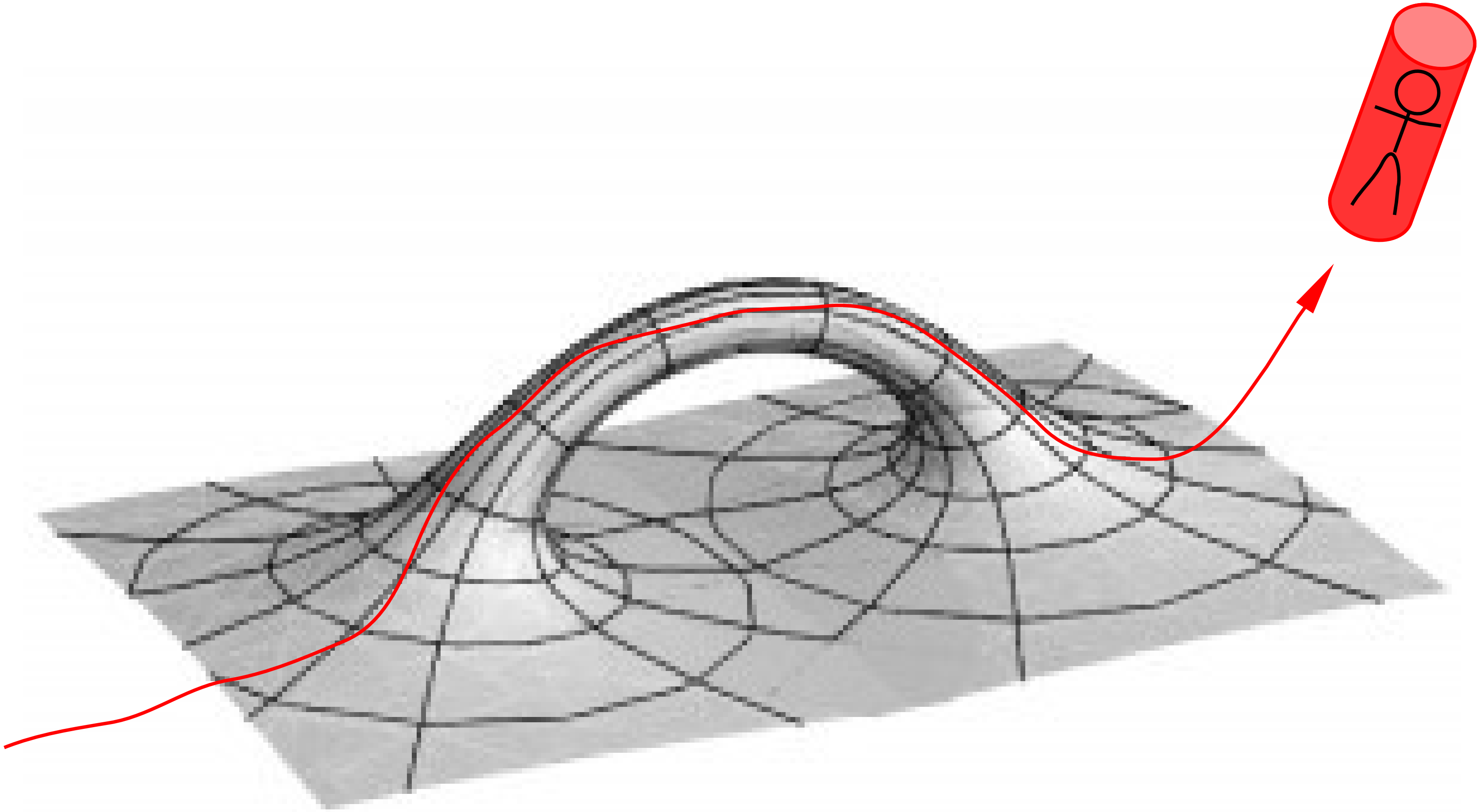
If there are no restrictions on $T_{\mu\nu}$,
Einstein's Equation has no content,
and *any* geometry you like could be a solution:

$$g_{\mu\nu} = ?$$

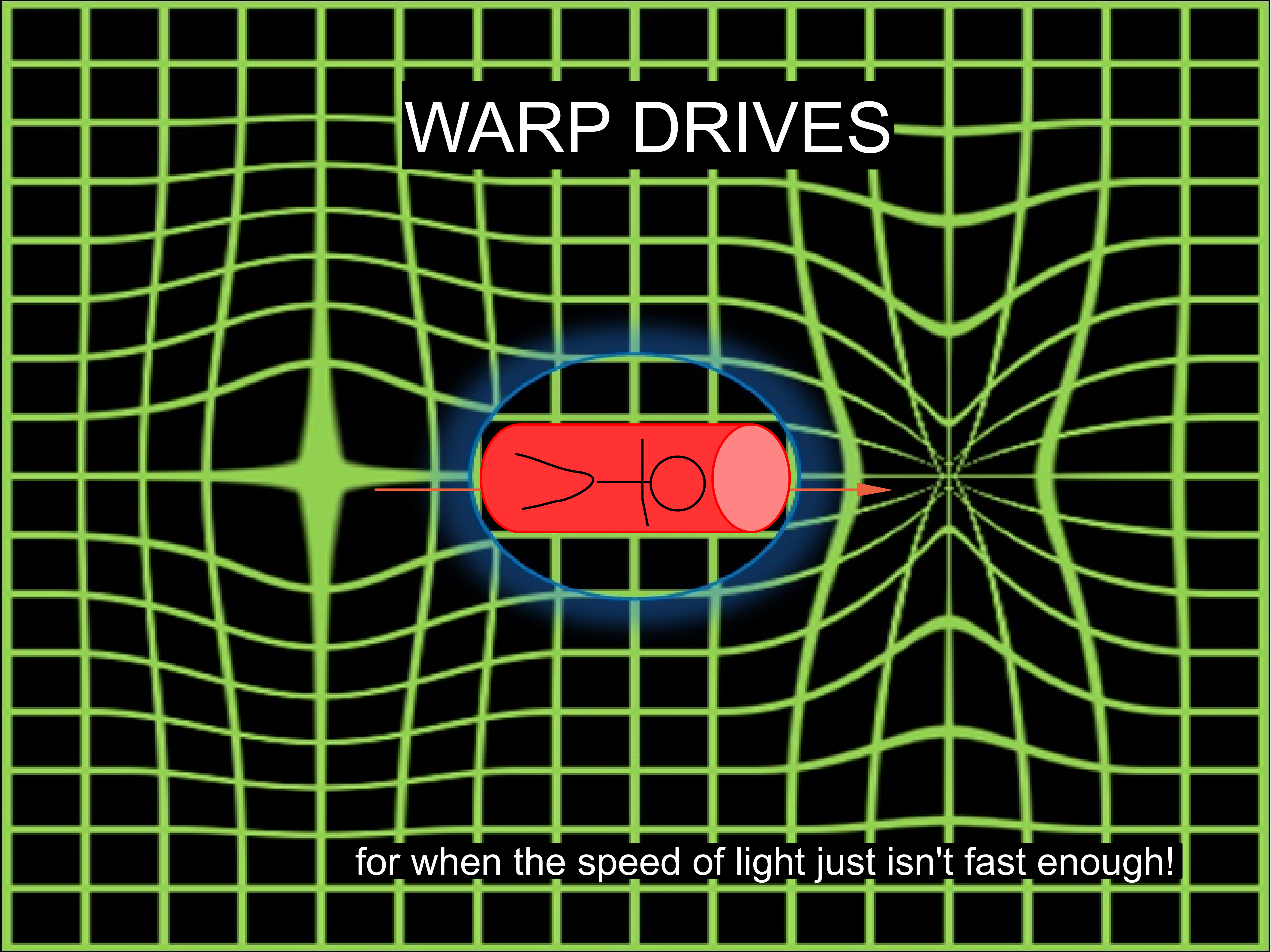
Many science fiction possibilities...

TRAVERSABLE WORMHOLES

for getting to another universe, or elsewhere in our own



WARP DRIVES



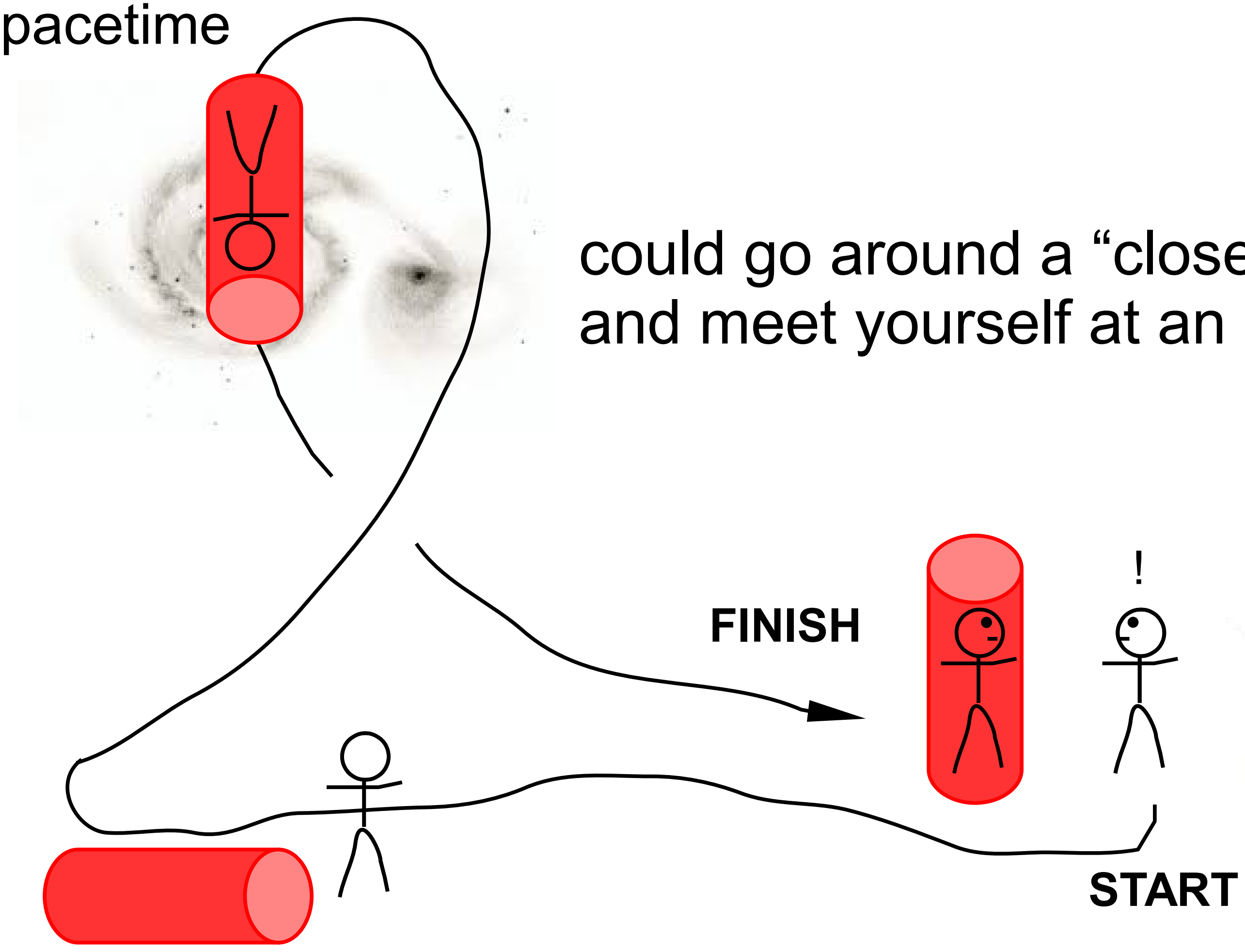
for when the speed of light just isn't fast enough!

and worst of all:

TIME MACHINES

for killing your grandfather before you are born
(and otherwise making a nuisance of yourself)

highly curved
spacetime



could go around a “closed timelike curve”
and meet yourself at an earlier time

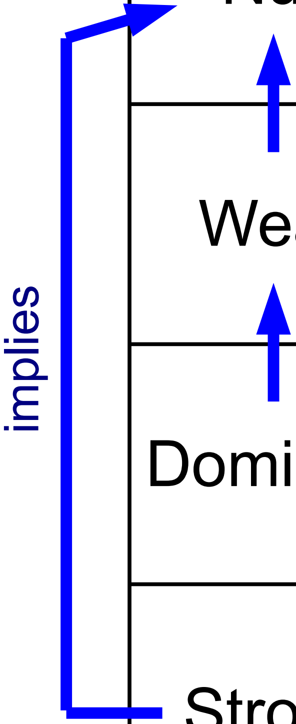
BUT ARE THESE CRAZY THINGS ACTUALLY POSSIBLE?

Not for reasonable classical matter fields.

All of them require exotic matter
which violates the “null energy condition”
satisfied by reasonable classical matter fields

Some Energy Conditions

k^μ : null vector $\qquad$ $t^\mu,\ u^\mu$: future timelike vectors



Condition	*this can't be negative:*	*perfect fluid*	*interpretation*
Null	$T_{\mu\nu}\, k^\mu k^\nu$	$\rho + p \geq 0$	null surfaces focus
Weak	$T_{\mu\nu}\, t^\mu\, t^\nu$	$\rho \geq 0$ $\rho + p \geq 0$	positive energy in any frame
Dominant	$T_{\mu\nu}\, t^\mu\, u^\nu$	$\rho \geq \lvert p \rvert$	energy can't go faster than light
Strong	$\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) t^\mu t^\nu$	$\rho + p \geq 0$ $\rho + 3p \geq 0$	timelike geodesics focus

Strong energy condition is violated for scalar fields with potential $V(\phi)$, e.g. inflation

All of these conditions are VIOLATED by quantum fields!

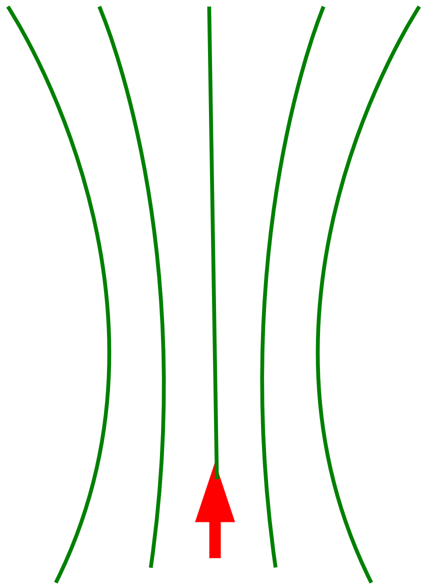
Quantum Energy Condition Violations

In QFT, ***all*** local energy conditions can be violated in certain states although **-** energy must be balanced by **+** energy elsewhere:
(**Klinkhamer 91**, **Folacci 92**, **Verch 00**, various papers by **Ford & Roman**...)

- Casimir effect (**Brown-Maclay 69**)
- moving mirrors (**Davies-Fulling 76, 77**)
- squeezed states (**Braunstein, in Morris-Thorne 88**)

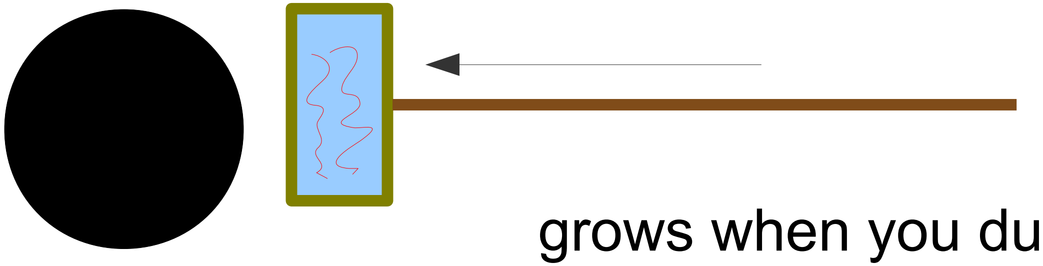
…and more

So can all these global results be circumvented?

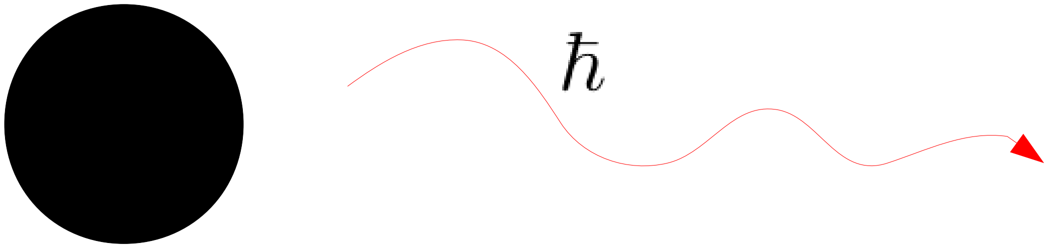


negative energy causes lightrays
to defocus, many GR proofs
require focussing

Black holes behave like thermodynamic systems (Bekenstein-Hawking)



grows when you dump matter in



shrinks as Hawking radiation is emitted

black holes have temperature, and energy, thus an entropy

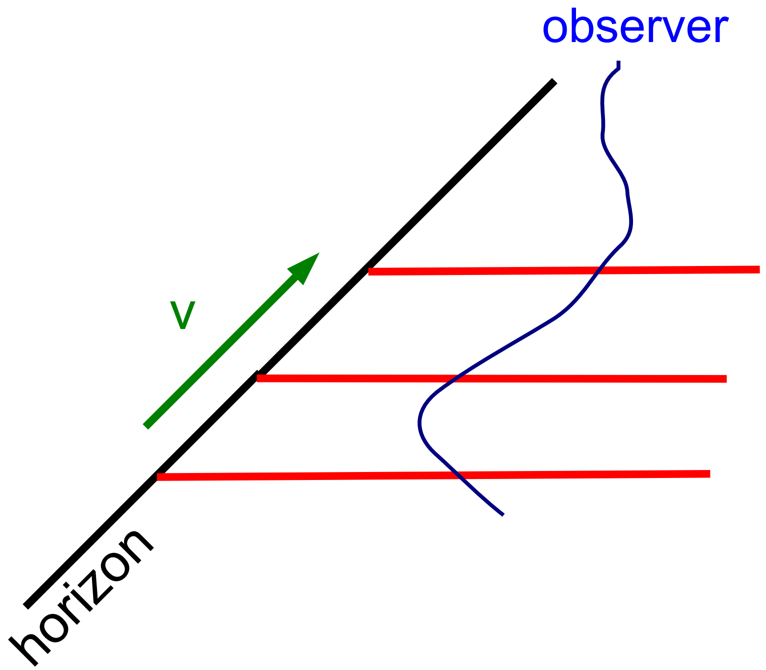
$$dE = TdS$$

proportional to the *area* of the horizon!

(also applies to other causal horizons e.g. de Sitter, Rindler)

Generalized Second Law

In semiclassical GR, the outside of a causal horizon is an OPEN system—
info can leave (but not enter).



But the generalized entropy

$$S_{\text{gen}} = \frac{A}{4G\hbar} + S_{\text{out}}$$

is increasing with time:

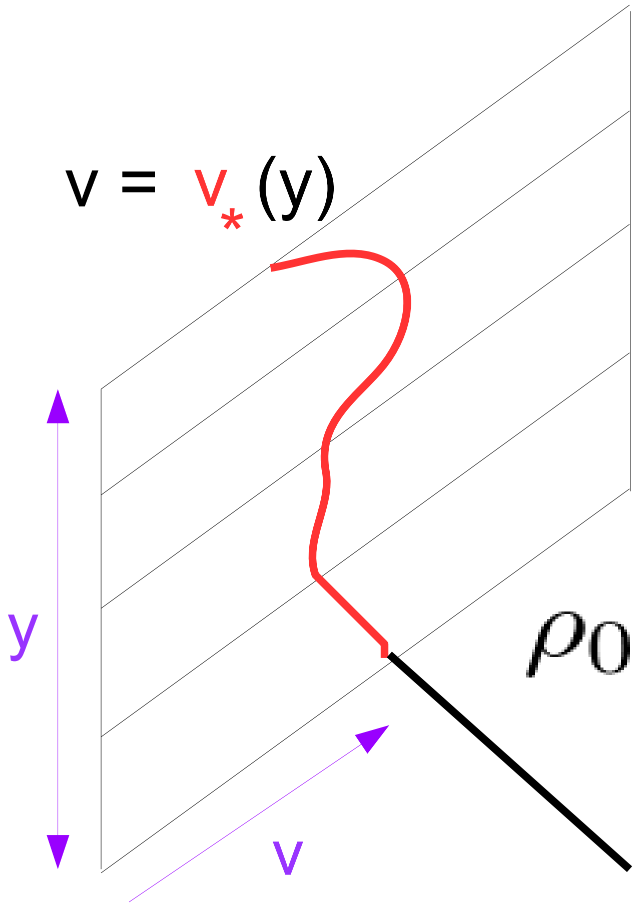
$$\frac{dS_{\text{gen}}}{d\text{v}} \geq 0$$

Generalized Second Law (GSL).

Conjectured by **Bekenstein '73** and **Hawking '75**
partial, limited proofs given in 80's – 00's, e.g. **Frolov & Page '93**

Proven for free fields (+ relevant interactions) in **Wall '11**
Extension @ linear order to all higher curvature gravity theories in **Wall '15**

Modular Hamiltonians on Null Surfaces



Key statement needed for proof is that for any wiggly slice on the horizon, the vacuum state outside takes the form of a thermal state:

$$\rho_0 = e^{-2\pi K/\hbar}$$

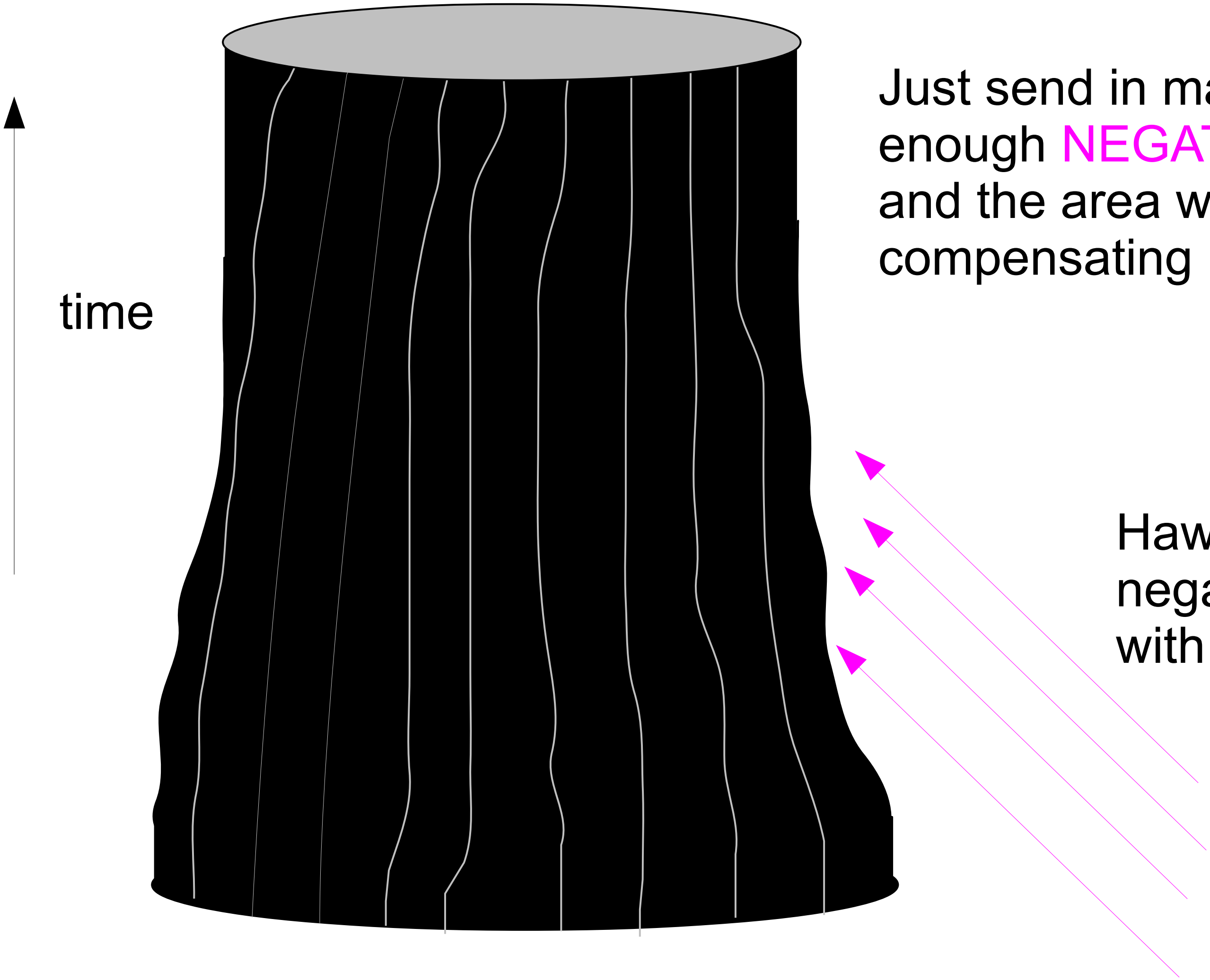
$$K = \int_{v_*}^{+\infty} (v - v_*) T_{vv} \, dv \, d^{D-2}y$$

Proven for free fields (+ relevant interactions) in **Wall 2011**
using monotonicity of relative entropy & lightfront quantization

Extension to general interacting case in **Casini, Teste, Torroba 2017**

implies: - GSL
- “Markov property” of vacuum,
- entanglement proof of a-theorem in D = 4

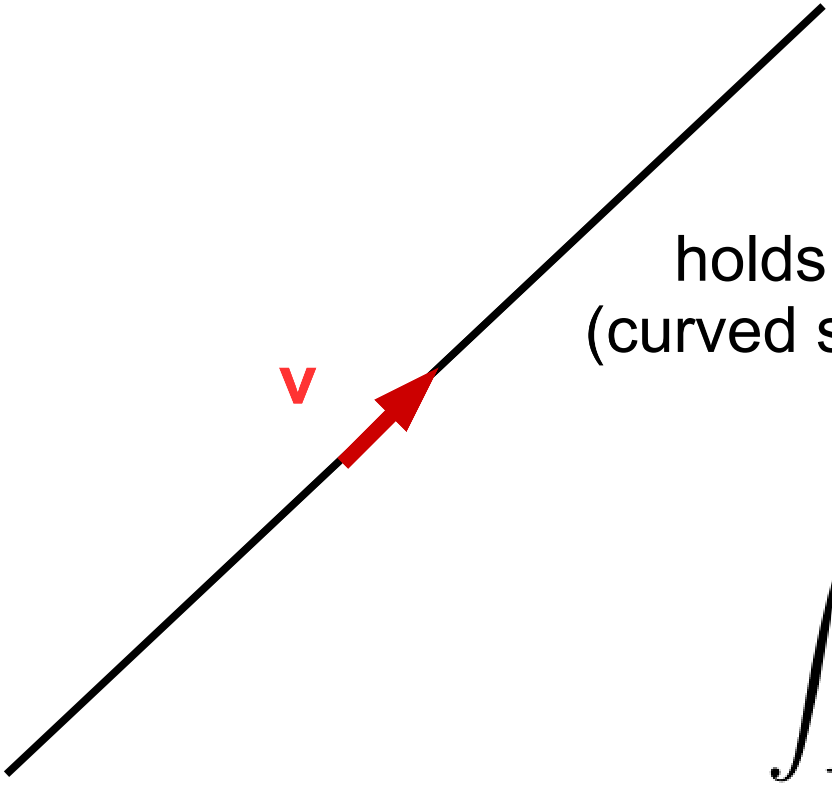
Suppose we have access to arbitrary negative energies. Then violating the Generalized Second Law (GSL) is easy.



Just send in matter which contains enough NEGATIVE ENERGY and the area will shrink without any compensating increase of entropy outside

Hawking radiation also has negative energy, but associated with *outgoing* entropy.

Energy & entropy of field theory must be intimately related



(Achronal) ANEC

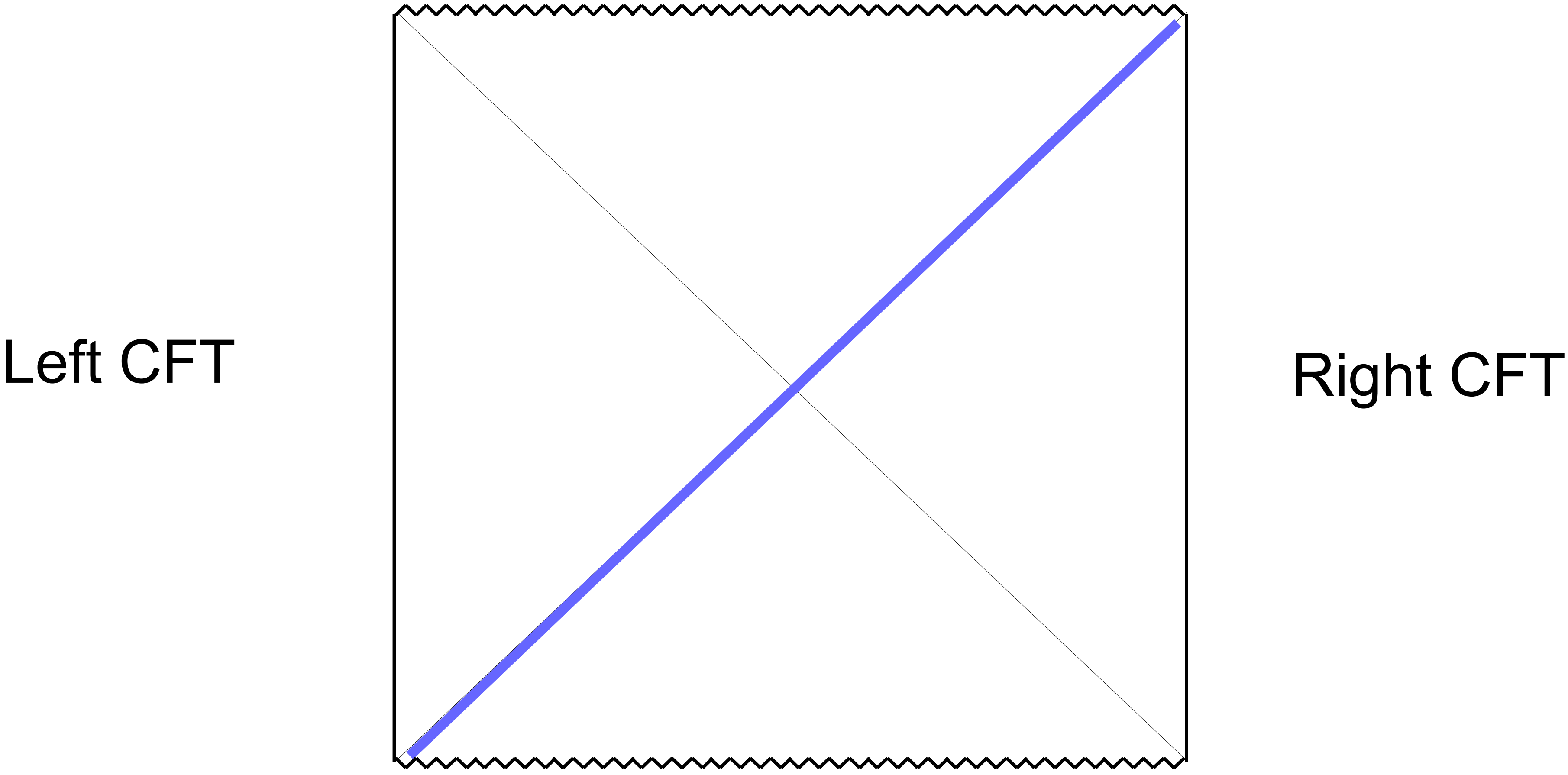
holds along infinite null geodesic
(curved spacetime: no 2 pts timelike separated,
Graham & Olum '07)

$$\int_{-\infty}^{\infty} T_{vv}\, dv \geq 0$$

Wall '10: GSL → ANEC
Wall '11: monotonicity of relative entropy → GSL (fields free in UV)

Faulkner, Leigh, Parrikar, Wang '16
monotonicity of relative entropy + perturbation thy for modular Hamiltonian
→ ANEC for arbitrary CFT's

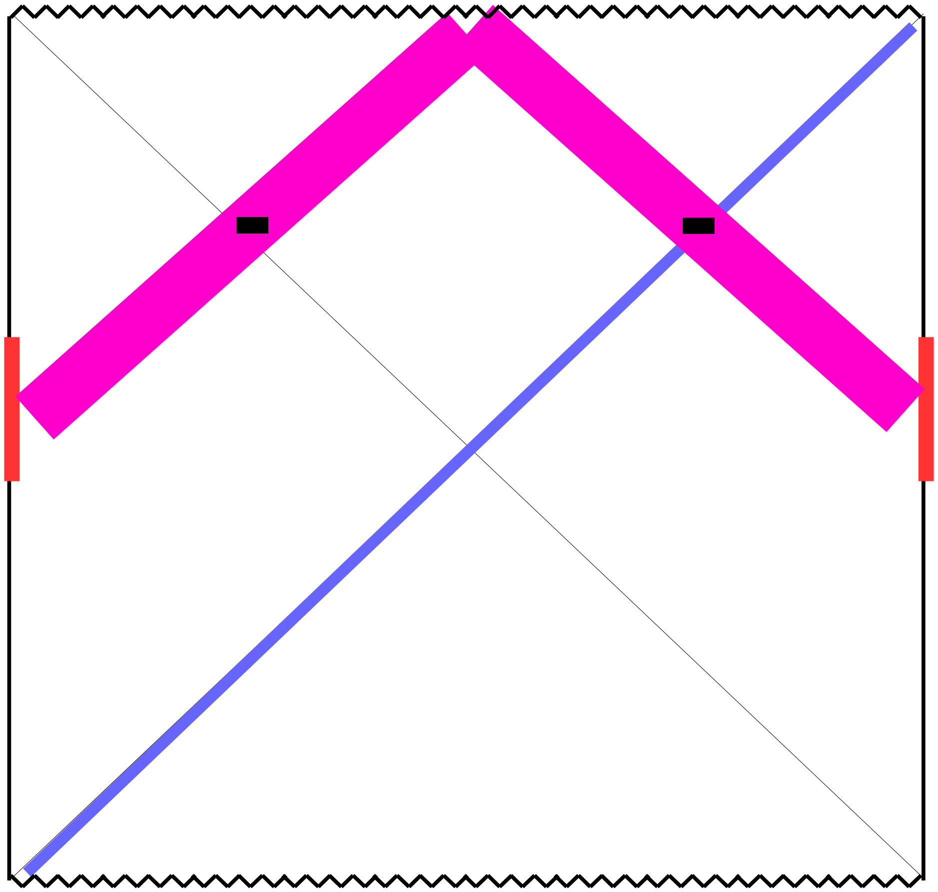
AdS/CFT: thermofield double dual to Einstein Rosen bridge



ANEC forbids traversable wormholes

Loophole: couple the CFT's to each other,

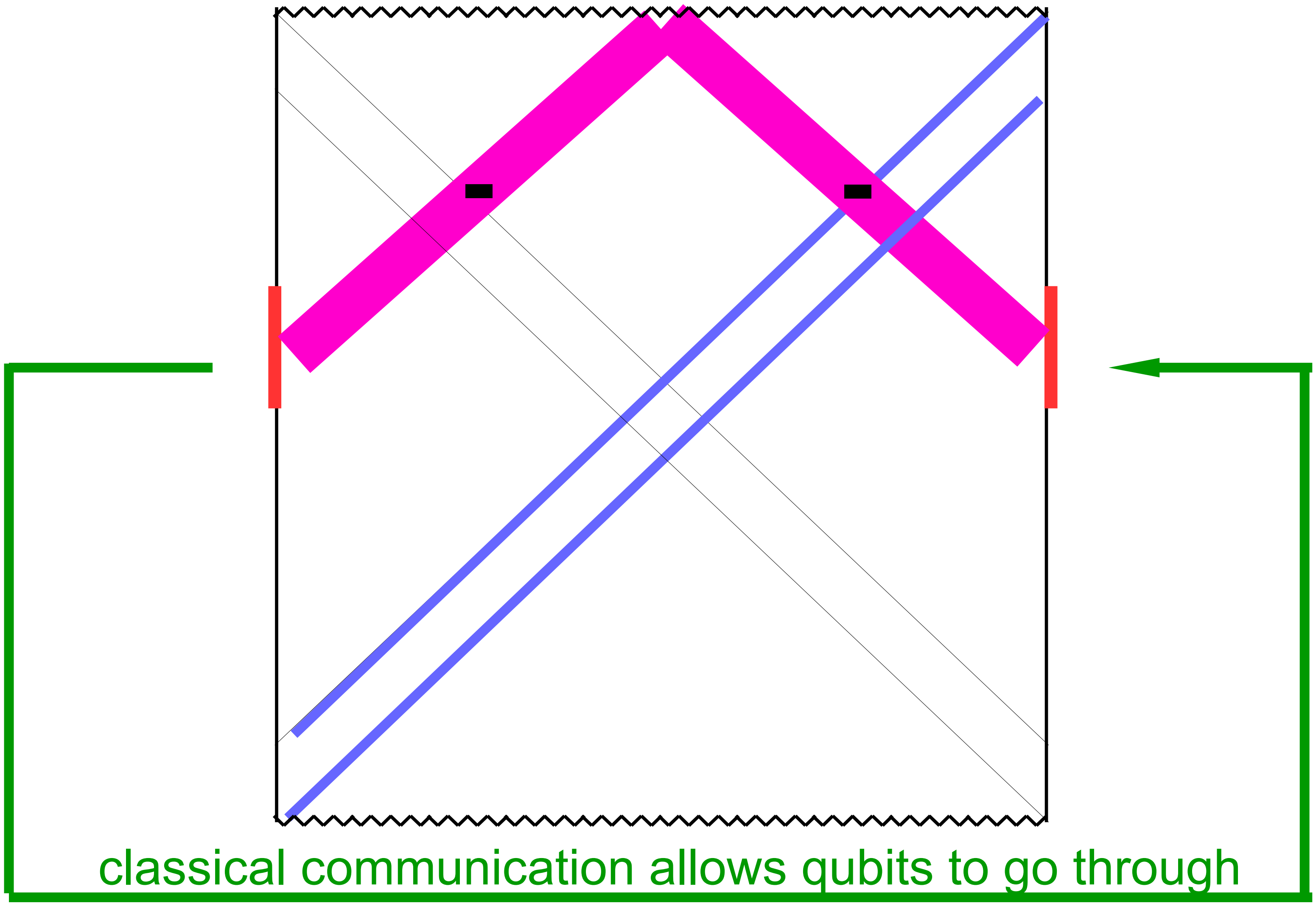
negative energy flux for one sign of coupling



$$H_{\rm int} = \int dt\, h(t)\, \mathcal{O}_L \mathcal{O}_R$$

Gao, Jafferis, Wall '16

Wormhole becomes traversable dual to quantum teleportation



classical communication allows qubits to go through

Gao, Jafferis, Wall '16; Maldacena, Stanford, Yang '17; Susskind, Zhao '17

From Classical
to Quantum

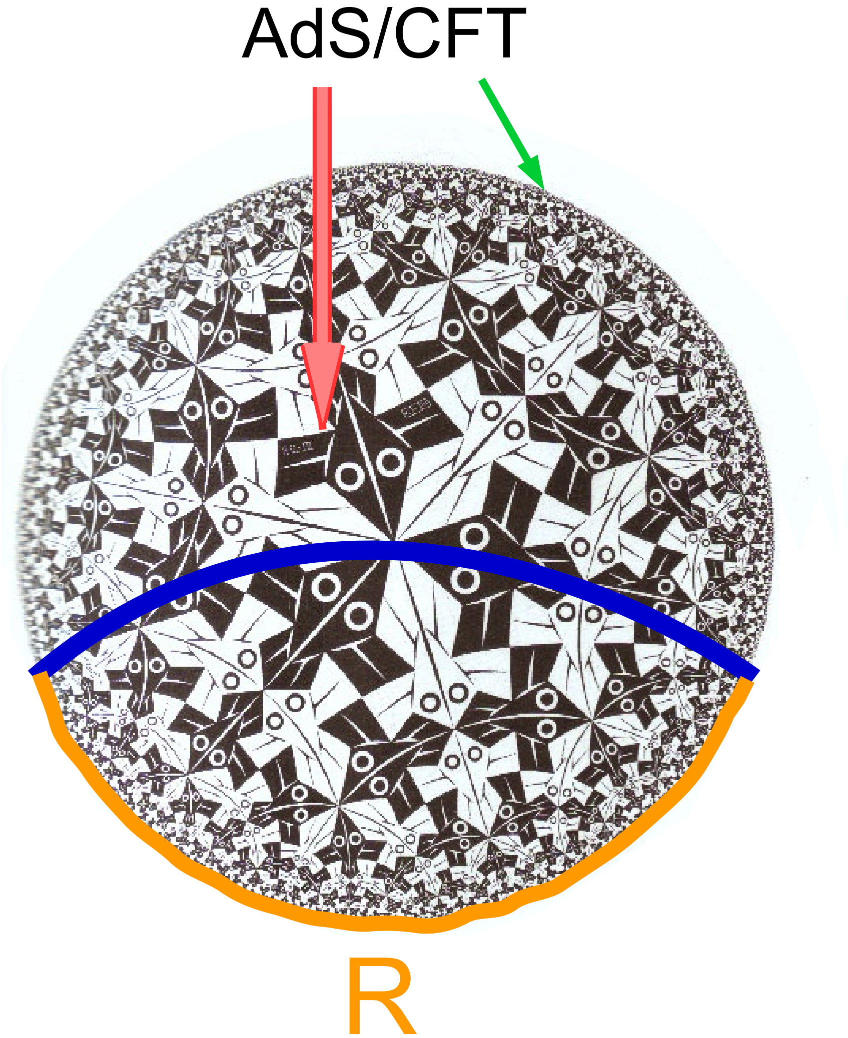
GSL suggests way to extend classical GR results to "semiclassical" situations involving quantum fields, (which violate classical energy conditions)

just replace the area with the generalized entropy!

$$A \to 4G\hbar S_{\mathrm{gen}}$$

Wall 2010: "The Generalized Second Law as a Quantum Singularity Theorem"

Example 1: Holographic Entanglement Entropy



Ryu-Takayangi '06
Hubeny-Rangamani-Takayanagi '07
valid for classical bulk (infinite N):

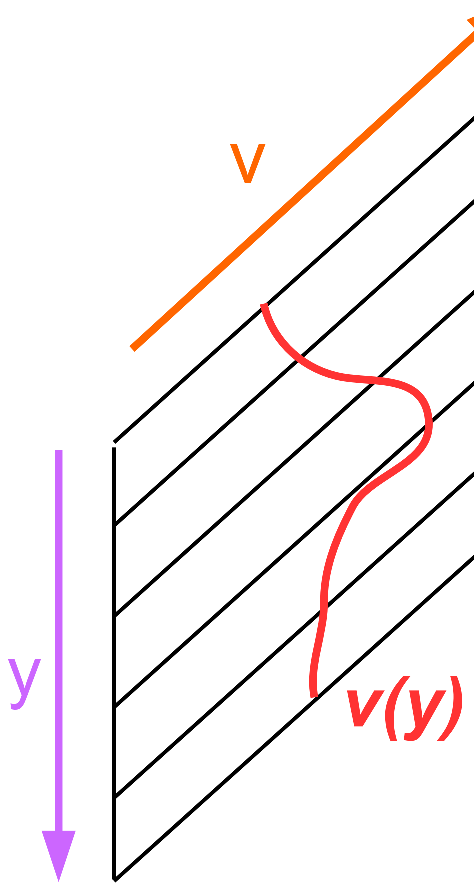
$$S(R) = \text{ext}\,\frac{A}{4G\hbar}$$

Engelhardt-Wall '14
(extending FLM '13)
valid at all orders in 1/N:

$$S(R) = \text{ext}\left(\frac{A}{4G\hbar} + S_{\text{bulk}}\right)$$

proven by **Dong & Lewkowycz '17**

Example 2: Quantum Focussing Condition



Classically, area focusses due to + null energy:

$$\frac{\delta}{\delta v(y)} \frac{\delta}{\delta v(y')} A \leq 0$$

[these fnl derivs are *densitized*, i.e. per unit area]

In semiclassical quantum gravity, natural to conjecture *generalized entropy* focusses:

$$\frac{\delta}{\delta v(y)} \frac{\delta}{\delta v(y')} S_{\text{gen}} \leq 0$$

Bousso, Fisher, Leichenauer, Wall '15

If we throw quantum fields across a nearly stationary surface, as G → 0, focussing of Area part is proportional to $T_{vv}\delta(y-y')$

We then get the *Quantum Null Energy Condition:*

$$\langle T_{vv} \rangle \geq \frac{\hbar}{2\pi} S''$$

(S'' is the local 2nd derivative of S_{out})

Quantum Null Energy Condition

$$\boxed{\langle T_{vv} \rangle \geq \frac{\hbar}{2\pi} S''}$$

Proofs (for general states unless stated otherwise):

- D = 2 conformal vacua (**Wall '11** “Testing the Generalized Second Law...”)
- free boson + relevant int. (**Bousso, Fisher, Koeller, Leichenauer, Wall '15**)
- holographic regime (**Koeller, Leichenauer '15**)
- *all* interacting D > 2 CFT's (w/ twist gap) + relevant int. (**Balakrishnan, Faulkner, Khandker, Wang '17)**

---

but even more astonishing, I've heard reports that the QNEC is ***saturated*** for all states in the interacting D > 2 case!

$$\langle T_{vv} \rangle = \frac{\hbar}{2\pi} S'' \text{!!!}$$

Leichenauer, Levine, Moghaddam '18 (holographic)

Chandrasekaran, Levine, Moghaddam, Faulkner, Balakrishnan (general CFT w/ twist gap, forthcoming)

From Global to Local Energy Conditions

based on arXiv:1701.03196:
“A Lower Bound on the Energy Density in Classical and Quantum Field Theories” Phys. Rev. Lett. **118**, 151601 (2017)

(non-gravitational)

Consider a classical field theory with **1** spatial direction and some “energy density” *T* whose integral is positive:

$$E = \int_{-\infty}^{+\infty} T\,dx \ \geq 0$$

$T(\phi_i,\, {\phi_i}',\, {\phi_i}'' \ldots)$ local function of fields & any # derivs. (includes any canonical momenta π_i)

[optional] may exist constraints (e.g. Gauss law):

$$C(\phi_i,\, {\phi_i}',\, {\phi_i}'' \ldots) = 0$$

Can we show T ≥ 0?

$$E = \int_{-\infty}^{+\infty} T\,dx \;\geq 0 \quad \xrightarrow{?} \quad T \geq 0$$

GLOBAL STABILITY

LOCAL ENERGY CONDITION

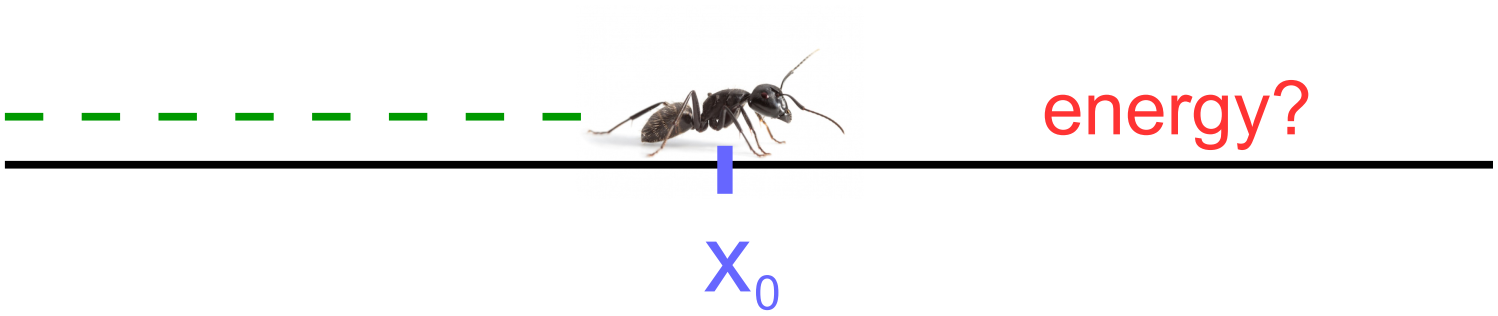
FALSE: Not true in general since we have the freedom to “improve” *T* by adding a total derivative M'

integral same if $M \to 0$ as $|x| \to \pm\infty$

Best we can hope for is $T + M' \geq 0$

WE WILL NOW PROVE THIS FOR *ALL* STABLE 1d FIELD THEORIES

PARABLE OF THE ANT



- ant travels from $x = -\infty$ to $x = x_0$

- she measures all fields ϕ_i along her path

- as a physicist, she knows formula for T (and C's).

- wants to predict lower bound on energy-yet-to-come:

$$M(x_0) \equiv \inf\left(\int_{x_0}^{+\infty} T\, dx \,\middle|\, \phi_i(x < x_0) \right)$$

$$M(x_0) \equiv \inf\left(\int_{x_0}^{+\infty} T\,dx \,\middle|\, \phi_i(x < x_0)\right)$$

1. M is FINITE, since E ≥ 0 implies:

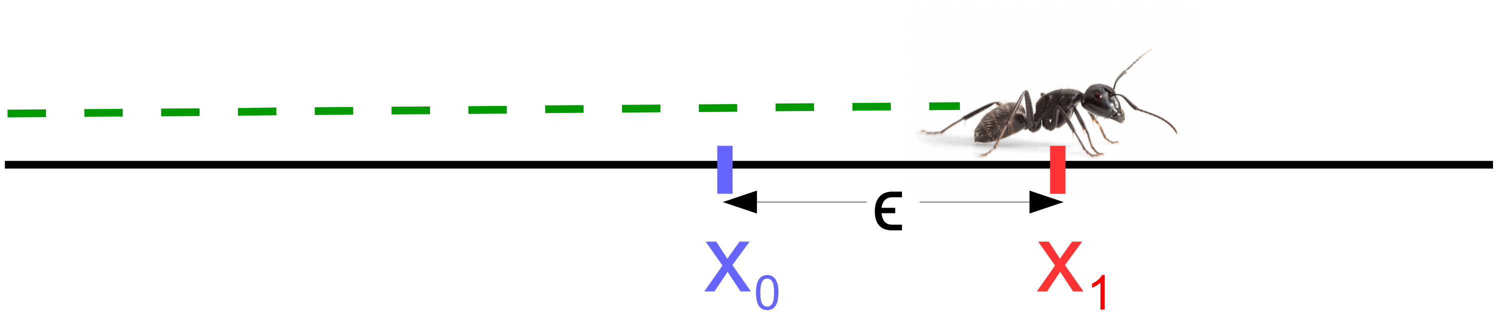
$$\int_{x_0}^{+\infty} T\,dx \geq -\int_{-\infty}^{x_0} T\,dx$$

but that's not the tightest lower bound, since

2. M is a LOCAL function of the fields & derivs

bound on energy depends only on neigborhood of x_0
since T, C are local functions

3. MONOTONICITY



as the ant crawls further, she learns MORE about the world, but learning more can only make lower bounds to go UP:

$$M(x_0) \;\le\; \inf\left(\int_{x_0}^{+\infty} T\,dx \,\middle|\, \phi_i(x < x_1) \right)$$

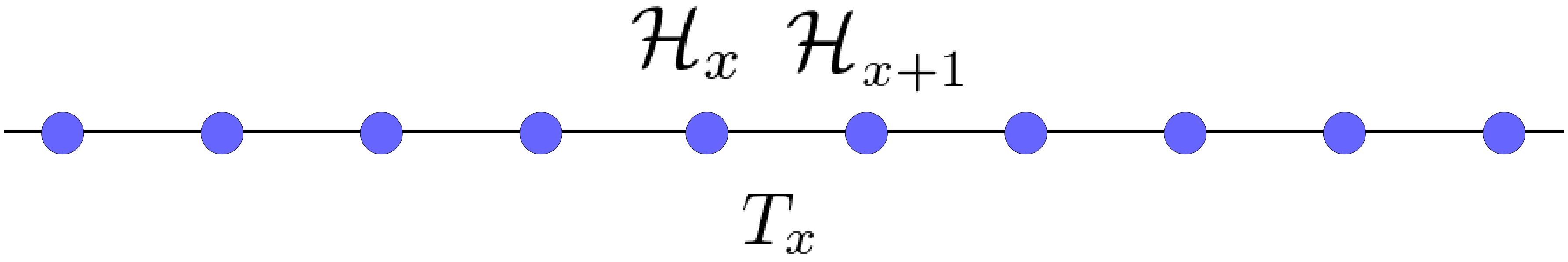
$$= \int_{x_0}^{x_1} T\,dx \;+\; \underbrace{\inf\left(\int_{x_1}^{+\infty} T\,dx \,\middle|\, \phi_i(x < x_1) \right)}_{M(x_1)}$$

hence $\epsilon(T + M') \ge 0$ in all states – a local energy condition

Quantum Energy Conditions

QM systems have new feature: entanglement

e.g. quantum spin system



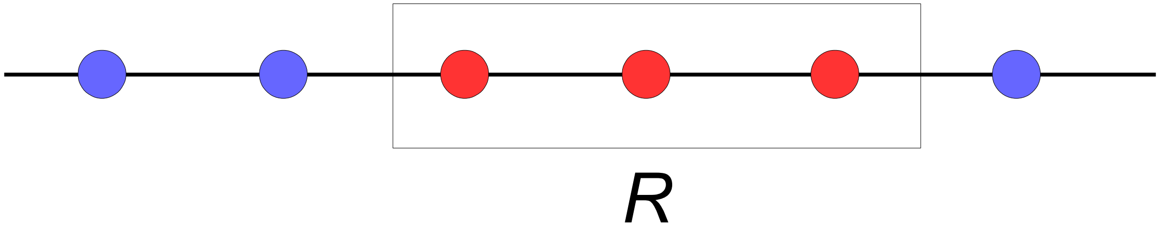
$T_x \in \mathcal{A}(H_x \otimes H_{x+1})$ nearest neighbor interactions

in general, cannot simultaneously diagonalize T's,
ground state is a compromise! -> vacuum entanglement

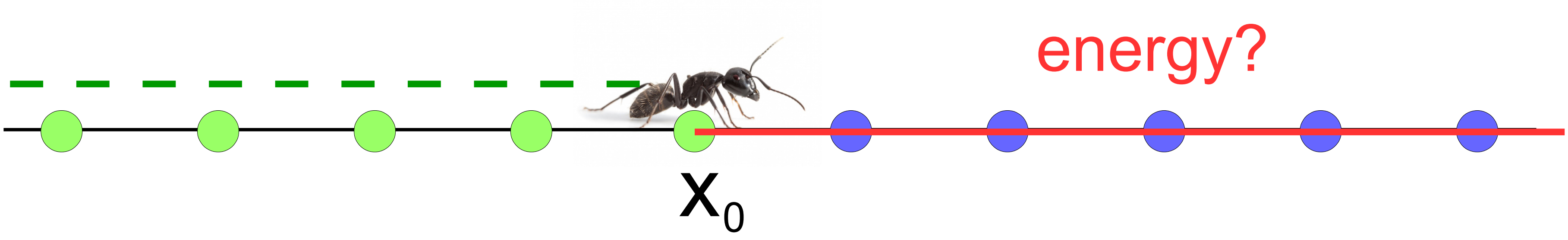
diagnostic: Entanglement Entropy

$$S = -\mathrm{tr}(\rho_R \ln \rho_R)$$

$$\rho_R = \mathrm{tr}_{\bar{R}}(|\Psi\rangle\langle\Psi|)$$



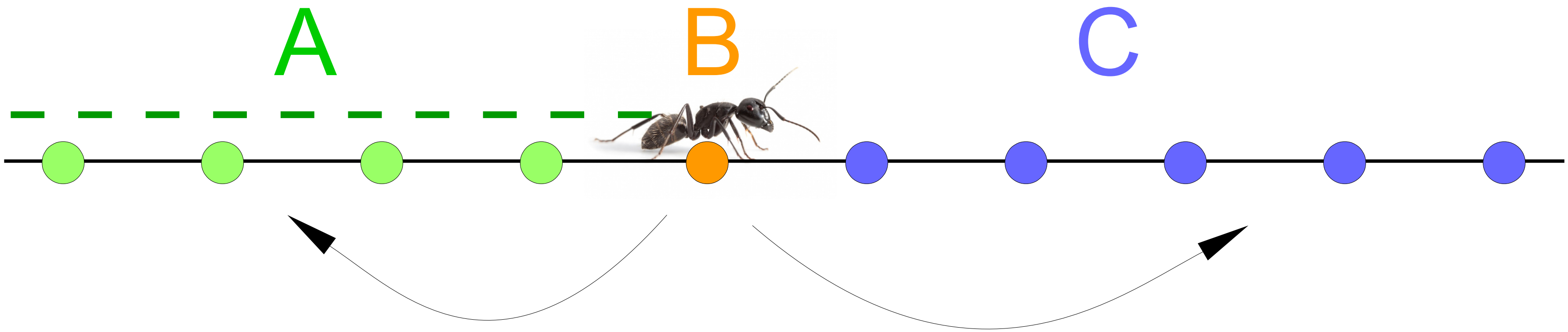
Repeat same arguments as in the classical case



$$M(x_0) \equiv \inf \left(\sum_{x=x_0}^{+\infty} \langle T_x \rangle \middle| \; \rho_{x<x_0} \right)$$

1. M is FINITE
~~2. M is a LOCAL function of fields~~ entanglement nonlocal
3. M is MONOTONIC: T + M' ≥ 0 (' is discrete deriv.)

Entanglement Monogamy: Nonlocal Constraint



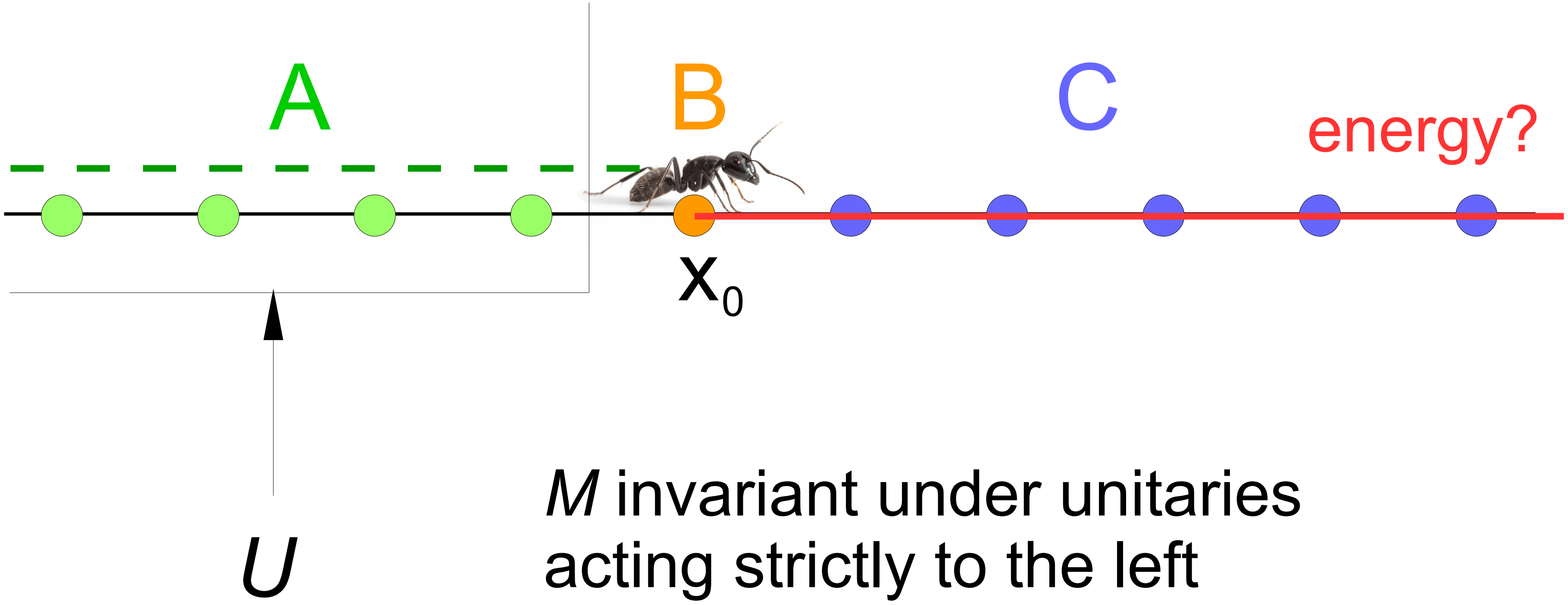
the more a qubit is entangled with the left...

...the less it can be entangled with the right

enforced by Strong Subadditivity:

$$S(AB) + S(BC) \geq S(A) + S(C)$$

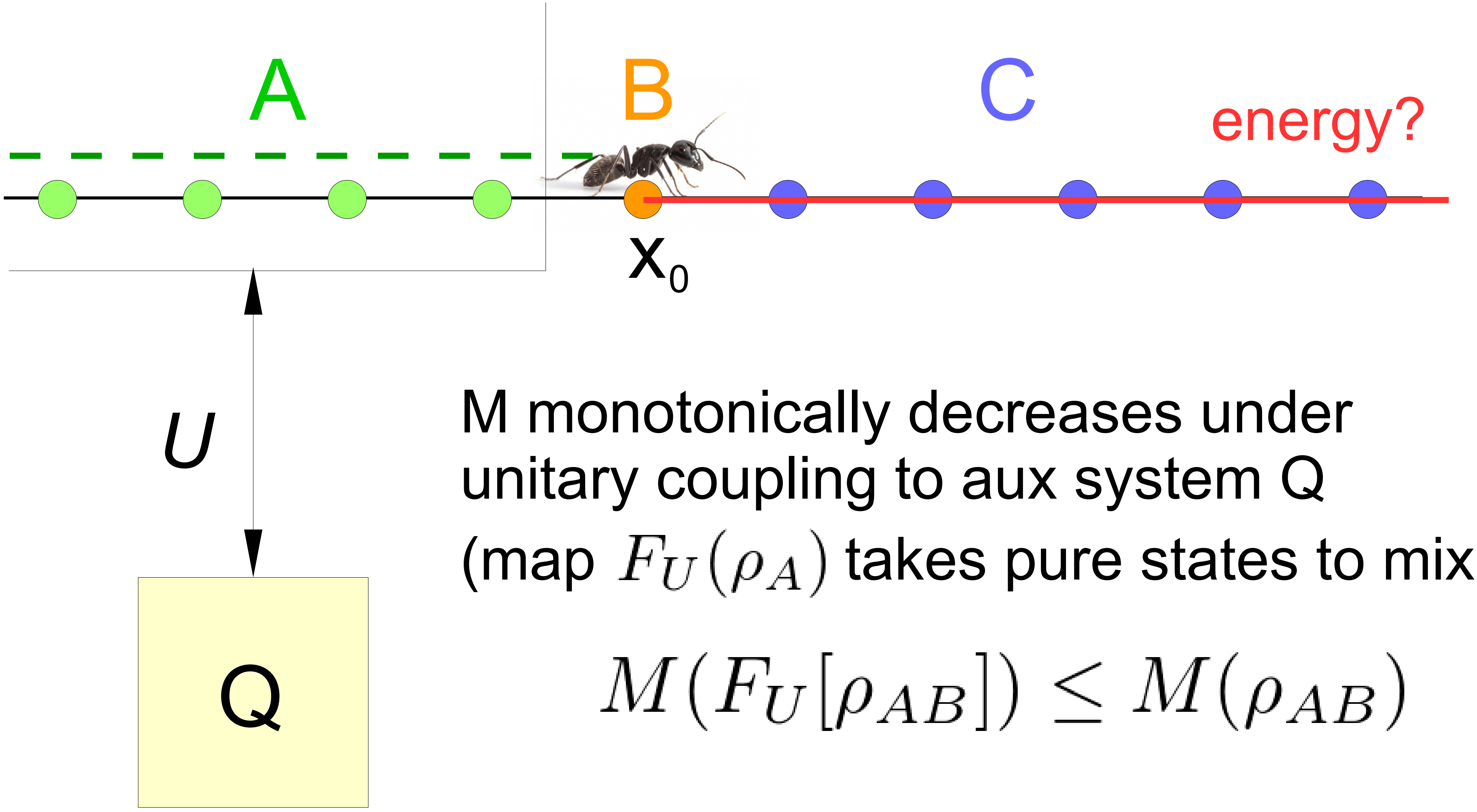
So M can be nonlocal, but still tightly constrained



M invariant under unitaries
acting strictly to the left
(i.e. depends only on *information* in A)

$$M[\rho_{AB}] = M[U_A\,\rho_{AB}\,U_A^\dagger]$$

So M can be nonlocal, but still tightly constrained



M monotonically decreases under unitary coupling to aux system Q

(map $F_U(\rho_A)$ takes pure states to mixed)

$$M(F_U[\rho_{AB}]) \leq M(\rho_{AB})$$

Works for M = -S' by Strong Subadditivity

$$S(A) - S(AB) \geq S(AQ) - S(ABQ)$$

Continuum QFT Limit

$$M(x_0) = \inf \left(\int_{x_0}^{+\infty} \langle T \rangle \, dx \,\middle|\, \rho_{x<x_0} \right)$$

$-S'$ is only (weight 1, monotonic) candidate I know for M, would imply QUANTUM ENERGY CONDITION:

$$\boxed{\langle T \rangle \geq \frac{\hbar}{2\pi} S''}$$

(c = 1)

coefficient fixed in relativistic field theory by vacuum thermality (Unruh effect):

$\rho_{x<x_0} \propto e^{-2\pi K/\hbar}$ where $K = \int_{x_0}^{+\infty} (x - x_0) \, T \, dx$

By independently boosting “x” and “E”, one can argue for several types of energy conditions in D = 1+1:

Quantum Null Energy Condition:

$$\langle T_{vv} \rangle \geq \frac{\hbar}{2\pi} \partial_v \partial_v S$$

v = null vector

Quantum Weak Energy Condition:

$$\langle T_{tt} \rangle \geq \frac{\hbar}{2\pi} \partial_x \partial_x S$$

t, *x* Minkowksi coordinates

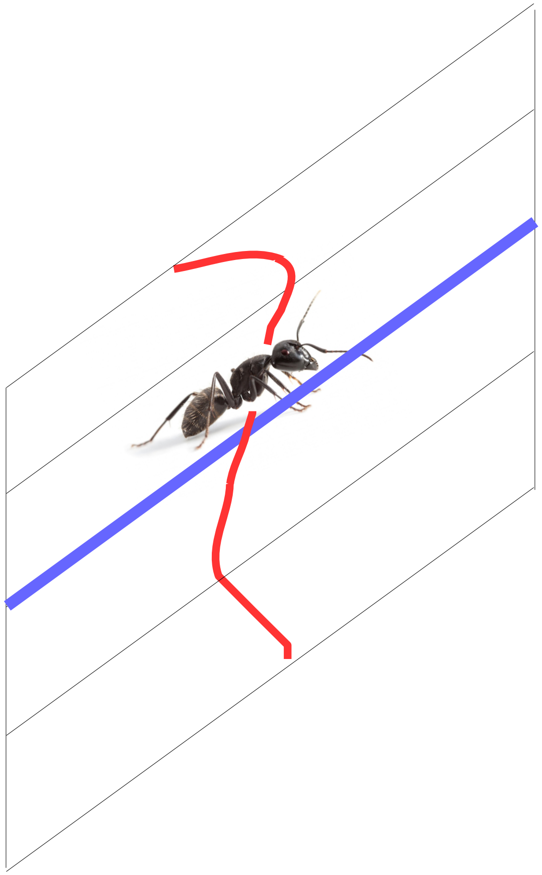
Quantum Dominant Energy Condition:

$$\partial_a^\perp = \epsilon_a^b \partial_a$$

$$\left(\langle T_{ab} \rangle - \frac{\hbar}{2\pi} \partial_a^\perp \partial_b^\perp S \right) t^a u^b \geq 0$$

t, *u* timelike vectors

Extension to Higher Dimensional QNEC?



Use ANEC along a single null line?

$$\int_{-\infty}^{\infty} T_{vv}\, dv \geq 0$$

$$\Big\downarrow ?$$

$$\langle T_{vv} \rangle \geq \frac{\hbar}{2\pi} S''$$

Conclusions:

- *Every* QFT satisfies *some* quantum energy condition
- relativistic field theories satisfy the QNEC
- for interacting D > 2 CFT's, QNEC is saturated!

- Energy and Information are intimately related, and their union will be fruitful for learning more about field theory.

END