The Casimir and Cardy Problems in $d = 4$ Quantum Field Theories

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Based in part on works that appeared with Closset, Di Pietro, Dumitrescu, Festuccia, Seiberg and on work in progress with Assel, Cassani, Di Pietro, Martelli...

First of all, it is of course a tremendous honor to receive the Philippe Meyer Prize, so I want to thank everybody who was involved.

In these slides there would be relatively little references to the literature – please ask me after or during the lecture if you are interested in more information.

Supersymmetric Quantum Field Theories continue to be interesting toy models for various phenomena in Quantum Field Theory.

The unifying theme behind recent developments is the study of Supersymmetric Quantum Field Theories in curved space.

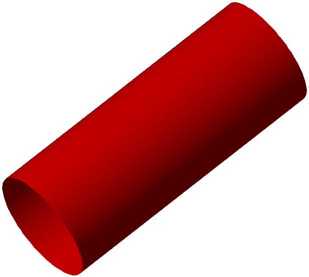
I would like to motivate the subject by considering two problems, which have been solved long ago for $d = 2$ theories.

Conformal Field Theories are of crucial importance in statistical physics, condensed matter physics, and high-energy physics.

Consider a CFT_2, with central charge c. Let us study the theory on the cylinder $\mathbb{S}^1 \times \mathbb{R}$. The ground state (Casimir) energy is

$$E_0 = -\frac{c}{12l} \ .$$

For a free boson it comes from $\sum n = -1/12$.



The general proof relies on the observation that $\mathbb{S}^1 \times \mathbb{R}$ is conformally equivalent to $\mathbb{R}^2$. The energy-momentum tensor transform with a Schwarzian derivative.

We can study the theory $\mathbb{S}^1 \times \mathbb{S}^1_\beta$ and consider the partition function for large β

$$\lim_{\beta\to\infty} Z(\beta) = e^{-E_0\beta} + \ldots = e^{\frac{c}{12}\beta} + \ldots.$$

By a modular transformation this can be related to the small β limit of the partition function. One finds the Cardy formula:

$$\lim_{\beta\to 0} Z(\beta) = e^{\frac{(2\pi)^2 c}{12\beta}} + \ldots$$

We can now consider a $d = 4$ CFT and put it on $\mathbb{S}^3 \times \mathbb{R}$, which is conformally flat. We can then compactify the Euclidean time direction to be $\mathbb{S}^1_\beta$. We can ask the same two questions:

- What is the large β limit? (Casimir)
- What is the small β limit? (Cardy)

The conformal anomalies of $d = 4$ CFT are known to be just a, c

$$T^\mu_\mu \sim aE_4 + cW^2$$

and these fix the analog of the Schwarzian derivative in $d = 4$. Following this idea one finds

$$E_0 = \frac{3}{4}a \ .$$

However, people have done explicit computations of the ground state energy E_0 and got different answers using different regularizations [Birrell, Davies ; Brown, Cassidy.....]

One should realize that E_0 is actually **unphysical** in $d = 4$. It depends on the regularization scheme. The point is that we can add

$$\delta S \sim b \int d^4x\sqrt{g}R^2 \ . \qquad T^\mu_\mu = ... + b\Box R$$

This can be viewed as a modification of the Schwarzian derivative. No such modification exists in $d = 2$.

This clearly modifies the ground state energy by $\delta E_0 \sim b/r$. The parameter b can be tuned to any desirable value.

One of the points below would be a proof that, for supersymmetric CFTs, the parameter b **disappears**! Therefore, the Casimir energy becomes physical again. And the ground state energy is not $\frac{3}{4}a$.

Consider the small β limit of a $d = 4$ CFT on $\mathbb{S}^3 \times \mathbb{S}^1_\beta$. We can understand this by a local action for the background fields in $d = 3$:

$$\lim_{\beta \to 0} Z_{\mathbb{S}^3 \times \mathbb{S}^1_\beta} = e^{\frac{a'}{\beta^3} l^3 + \frac{b'}{\beta} l + \dots}$$

$$S_{d=3} = \int d^3x \sqrt{g} \left(a' T^3 + b' TR + \dots \right) \ ,$$

where R is the Ricci scalar. This is the usual high-temperature expansion.

In examples we find that a', b' are not given by any combination of anomalies. For example, a', b' depend on the coupling constants.

Below we will see that in supersymmetric theories $a' = 0$ and

$$b' \sim (a - c)$$

More precisely, we will see that in supersymmetric theories

$$\lim_{\beta\to 0} Z(\beta) = \lim_{\beta\to 0} Tr_{\mathcal{H}}((-1)^F e^{-\beta H}) = e^{\frac{(4\pi)^2}{3\beta}(c-a)}$$

This is very much reminiscent of the Cardy formula. Does this suggest some kind of generalized modular invariance in $d = 4$? maybe...

Summary: The Casimir and Cardy problem appear to be extremely natural in supersymmetric theories in $d = 4$. To study them we need to understand supersymmetry in curved space. This is a fascinating story by itself, which has lot of other applications.

Supersymmetric Field Theories have a conserved charge Q_α such that it squares to a translation

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^{\mu}_{\alpha\dot{\alpha}} P_\mu \ .$$

Supersymmetry transformations are generated by

$$\delta \equiv \zeta^\alpha Q_\alpha + \bar{\zeta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}$$

where ζ_α is a **constant** spinor

$$\partial_\mu \zeta_\alpha = 0 \ .$$

If we take a supersymmetric theory and just naively replace

$$\partial_\mu \to \nabla_\mu$$

we will have no leftover supersymmetry in curved space.

This is unless there are global solutions to

$$\nabla_\mu \zeta_\alpha = 0 \ .$$

To preserve some supersymmetry in curved space, we need to turn on additional background fields and possibly add non-minimal couplings to gravity.

Here is how to do it systematically:

We start from $\mathcal{N}=1$ in $\mathbb{R}^4$. The energy-momentum tensor resides in a multiplet with bosonic components

$$\left(j_\mu^R, T_{\mu\nu}, F_{\mu\nu}\right) \ ,$$

where $\partial^\mu j_\mu^R = 0$, $dF = 0$, and the energy-momentum tensor is symmetric and conserved. This is a 12+12 multiplet. It can be nicely packed in superspace

$$\bar{D}^{\dot\alpha}\mathcal{J}_{\alpha\dot\alpha} = D_\alpha\chi \ , \qquad \bar{D}\chi = 0 \ , \qquad D^\alpha\chi_\alpha = \bar{D}_{\dot\alpha}\bar{\chi}^{\dot\alpha} \ .$$

[ZK,Seiberg]

We couple the R-multiplet to background fields:

$T_{\mu\nu}$	$g_{\mu\nu}$
j_μ^R	A_μ^R
$F_{\mu\nu}$	$\epsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$

We emphasize that the fields $g_{\mu\nu}, A_\mu^R, B_{\mu\nu}$ are not path integrated over.

For a general choice of these background fields, the coupling to curved space breaks supersymmetry. The (g, A^R, B) multiplet is known as the "new-minimal" multiplet.
[Sohnius, West]

So how do we find choices of (g, A^R, B) for which there is an unbroken *rigid* supersymmetry?
A manifold would admit unbroken rigid supersymmetry if there is a spinor that solves [Festuccia,Seiberg]

$$\delta\psi_{\mu\alpha} = 0$$

This leads to a differential equation for ζ_α that takes the form

$$(\nabla_\mu - A^R_\mu)\zeta = \sigma_{\mu\nu}\epsilon^{\nu\alpha\beta\gamma}\partial_\alpha B_{\beta\gamma}\zeta \ .$$

and a similar equation for $\bar{\zeta}$.

Suppose we consider a theory that has a global symmetry group G. The bosonic operators in the supersymmetric current multiplet are

$$(J^G, j_\mu^G) \ .$$

We can couple them to background fields,

j_μ^G	A_μ^G
J^G	D^G

Again, only for certain choices of A_μ^G, D^G one may preserve a rigid supercharge. A_μ^G can be viewed as a connection for a G-bundle over $\mathcal{M}_4$.

It turns out that a necessary and sufficient condition to preserve at least one supercharge is that $\mathcal{M}_4$ is *Hermitian* and the G-bundle is *holomorphic*. [Closset, Dumitrescu,Festuccia,ZK, Seiberg]

A particularly interesting case to consider is

$$\mathbb{S}^3 \times \mathbb{S}^1$$

This is a Hermitian manifold, and there is a two-complex-dimensional moduli space of complex structures. One can also introduce holomorphic gauge bundles.

Let us specify a four-manifold $\mathcal{M}_4$ with some complex structure $J^2 = -1$, a Hermitian metric $g_{i\bar{j}}$, and some holomorphic G-bundle A_μ^G. So we have

$$Z_{\mathcal{M}_4}[J_i^j, \overline{J}_{\bar{i}}^{\bar{j}}, g_{i\bar{j}}, A_\mu^G \ldots]$$

The $\cdots$ stand for additional parameters, e.g. coupling constants.

The Partition Function

Several general properties of the partition function $Z_{\mathcal{M}_4}[J_i^j, \overline{J}_{\bar{i}}^{\bar{j}}, g_{i\bar{j}}, A_\mu^G, ...]$: [Closset, Dumitrescu, Festuccia, ZK]

- Given the complex structure $J^2 = -1$, the partition function is *independent* of the Hermitian metric $g_{i\bar{j}}$.
- The dependence on the complex structure moduli is *holomorphic*.
- The partition function depends *holomorphically* on the moduli of the holomorphic G-bundle.
- The partition function is independent of small variations of the coupling constants!

Let us therefore consider $\mathbb{S}^3 \times \mathbb{S}^1$ in more detail. We can choose the round metric, which is Hermitian with respect to the complex structure

$$ds^2 = d\theta^2 + (d\mathbb{S}^3)^2 \ .$$

To preserve supersymmetry we need to turn on some flux of $B_{\mu\nu}$ through the $\mathbb{S}^3$ as well as to turn on $A^R = \frac{1}{r} d\theta$.

There are four supercharges, which transform under $SU(2)_L \times SU(2)_R$ in

$$(1/2,0) \oplus (1/2,0)$$

The superalgebra is

$$\{Q_\alpha, Q^\dagger_{\dot\alpha}\} = \sigma^0_{\alpha\dot\alpha}(H - \frac{1}{r}R) + \frac{1}{r}\sigma^i_{\alpha\dot\alpha}J^i_L$$

In the limit of $r \to \infty$ we recover the flat space algebra. Let us specialize to one particular supercharge Q_2,

$$\{Q_2, Q^\dagger_{\dot 2}\} = H - \frac{1}{r}R + \frac{1}{r}J^3_L$$

We can interpret the partition function as a trace over the Hlibert space, $\mathcal{H}$, on $\mathbb{S}^3$,

$$Z(\beta) = Tr_{\mathcal{H}}((-1)^F e^{-\beta H})$$

β can be identified with one of the complex structure moduli of $\mathbb{S}^3 \times \mathbb{S}^1$. It is just the radius of $\mathbb{S}^1$ if we normalize the $\mathbb{S}^3$ to have unit radius.

From the superalgebra we see that we only receive contributions from states on $\mathbb{S}^3$ that satisfy

$$H - \frac{1}{r}R + \frac{1}{r}J_L^3 = 0 \ .$$

These are in short representations.

We start from the Supersymmetric Cardy problem. Suppose we take β to be very small. Then, we get a local action in $d = 3$ (at least as far as negative powers of β are concerned). We therefore need to supersymmetrize the $3d$ action on $\mathbb{S}^3$

$$S_{d=3} = \int d^3x\sqrt{g}\left(a' T^3 + b' TR + ...\right) \ ,$$

(where R is the Ricci scalar) such that it enjoys $\mathcal{N} = 2$ supersymmetry.

$$S_{d=3} = \int d^3x\sqrt{g}\left(a' T^3 + b' TR + ...\right) \ ,$$

- $a' = 0$ because the cosmological constant is not compatible with new-minimal supergravity.
- The supersymmetric Einstein-Hilbert term is in the same multiplet as a Chern-Simons term $A \wedge dG$ (G being the KK graviphoton). The coefficient of a Chern-Simons term cannot depend on coupling constants. We can compute it in free field theory. This gives $b' \sim (a - c)$. Therefore, the coefficient of R is protected and we can make a prediction about the asymptotics of the number of states on $\mathbb{S}^3$ [Di Pietro, ZK]:

$$\lim_{\beta\to 0} Tr_{\mathcal{H}}((-1)^F e^{-\beta H}) = e^{\frac{(4\pi)^2}{3\beta}(c-a)}$$

Let us now consider the opposite, Casimir, limit, i.e. $\beta \to \infty$.

We explained that it is unphysical in the general case. Let us prove that it is universal once we add supersymmetry.

In the limit $\beta \to \infty$ it is natural to reduce on the three-sphere in $\mathbb{S}^3 \times \mathbb{R}$. Then, we need to compute the ground state energy in Quantum Mechanics.

The Quantum Mechanics that we get has four supercharges and R-symmetry group $SU(2) \times U(1)$. The algebra contains for example

$$\{Q_2, Q_2^\dagger\} = H - \frac{1}{r}R + \frac{1}{r}J_L^3$$

Clearly the vacuum has $J_L^3|VAC\rangle = 0$, for otherwise, it would not be unique.

If we just had QM with finitely many degrees of freedom we could not fix E_0 because we can add a normal ordering constant to $H \to H + c/r$ and $R \to R + c$

Shifting the R-charge by a constant $R \to R + c$ corresponds to adding a Chern-Simons term in Quantum Mechanics

$$\delta S = c \int dt A_0^R$$

However, the allowed counter-terms must descend from counter-terms in four dimensions. The only possible four-dimensional term from which it could descend is

$$\int d^4x\sqrt{g}\; A_\mu^R \epsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}$$

which gives a dependence on r which is different than what we need.

This therefore proves that the Casimir energy on $\mathbb{S}^3$ is physical in supersymmetric theories.

The relation to the Chern-Simons term in Quantum Mechanics shows that the Casimir energy is independent of coupling constants.

We can thus compute it in free field theory. The computation is very subtle. A very preliminary conclusion is

$$E_0 = \frac{4}{27}(a + 3c)$$

but stay tuned.

In AdS/CFT, E_0 is interpreted as the (ADM) mass of AdS_5. If one is not careful about regularization, the ADM mass of AdS_5 is not meaningful (despite various "computations" of it in the literature).

But our discussion above shows that with a supersymmetric regularization it should be meaningful. It would be interesting to compute it and compare with field theory predictions. In fact, a lot of our discussion should be possible to recast in the language of holographic renormalization.

Another obvious extension would be to compute the Casimir energy on $\mathcal{M}_3 \times \mathbb{R}$ with arbitrary $\mathcal{M}_3$ that is Seifert. This looks doable.

Thank you for your attention